Contracting with Externality and Asymmetric Information

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Abstract

This paper analyzes the strategic situation with one principal and multiple agents, where the principal offers each agent a bilateral contract in order to purchase the share of property held by the agent. The agents have private information about their own types; in addition, there exists externality among the agents. I consider two types of contracts, single-price and contingent, and fully characterize the optimal contracts under certain conditions. A discussion on efficiency and an extension to general mechanism design are also provided.

Keywords: Contract Theory, Externality, Asymmetric Information, Contingent Contract, Mechanism Design

1 Introduction

In a strategic situation where a principal tries to reach agreement with a group of agents, it has been thoroughly studied what optimal contracts look like when the principal is able to bargain with each of the agents independently, i.e. the principal essentially solves a series of isolated bargaining problems. However, there are a wide range of cases under which the outcome from one contract would affect incentives for another, even if the principal can commit to the contract terms in each. In other words, when the action of some agent would affect others' payoffs, externalities arise from the bargaining process which need to be taken into consideration in the contract. For instance, before a court session involving multiple plaintiffs, from time to time the defendant would like to reach private agreements with all the plaintiffs for withdrawal of the accusation. Since the legal expense is usually shared (though not necessarily evenly) by the plaintiffs, it is crucial for the defendant to realize that the compensation offered to each plaintiff would greatly depend on the number of plaintiffs to be dealt with. This is an example where positive externalities among agents would force the principal to offer a higher stake to each agent than she needs to when bargaining with only one. On the other hand, in Cournot competition for example, the fact that more existing firms in the market would reduce profits for each makes it easier for one firm to acquire another, reflecting negative externalities at work.

Apart from externalities, another factor that may alter the contract design is that information is usually asymmetric between the principal and the agents, and even among agents. In most cases it is the distribution of agents' types, not the realization of types, that is known to both the principal and the agents (for the agents, they typically know their own types, but are not sure about others'). Such knowledge, or beliefs to be more precise, can be formed by past experience or through market surveys. The effect of asymmetric information here is two-fold: on one hand, the principle would have to pay informational rent to the agents due to the possibility of facing high-typed agents; on the other hand, information asymmetry *among* agents would reduce their bargaining power, compared to an otherwise identical case of full information. It is quite conceivable that in the optimal contract, which may well be contingent instead of independent as in the case without externality, there would be payments in the contract terms that will never occur in equilibrium. They would serve solely as a tool for manipulation of incentives.

Admittedly, the assumption of common prior belief and common knowledge of such belief is rather strong, but as the analysis in this paper would demonstrate, the implications of the model under this assumption is without loss of strong economic intuition. A model with this assumption relaxed, at the same time, would probably prove to be another interesting research topic.

This paper is organized in the following way. Section 2 provides a brief literature review on related topics. Section 3 introduces a benchmark model with single-price contracts, and Section 4 analyzes contingent contracts. In both sections 3 and 4, the optimal contract is explicitly derived under some regularity conditions. Section 5 generalizes the contract form to a class of commitment schemes and identifies the optimal contract form. Section 6 concludes.

2 Related Literature

Literature about contracting with externalities are widely spread in the fields of applied theory, mechanism design, industrial organization, and law. Segal (1999)[13]'s paper "Contracting with Externalities" provides a thorough survey, and to a large extent a unified theory, on the optimal contract as well as the potential inefficiency that follows in the presence of multilateral externalities and full information. Under a considerably general setting, he identifies sources of arising inefficiencies such as the principal lacking commitment power and the existence of noise in the execution of the mechanism. Segal (2003)[14] pushes the analysis one step further to discussing the effect of coordination and discrimination on the aggregate trade and efficiency, and it turns out that such effects would greatly depend on whether externalities are increasing or decreasing with the volume of trade. Moller (2007)[11] extends this model to endogenize the timing in sequential offering and Bernstein and Winter (2011)[2] takes the "divide and conquer" idea to characterize the optimal contracting scheme under full information and heterogeneous externalities.

Along the line of research on mechanism design, externalities and asymmetric information are closedly related to the class of environment with endogenous or type-dependent outside options. In other words, an agent's utility from rejecting the principal's offer would be affected by both his own type and other agents' decisions. In Jehiel et al. (1996)[8], the principal is trying to sell a single indivisible good, and externalities take the particular form of a matrix $[\alpha_{ij}]$ whose (i, j) entry represents externality imposed on agent j by a sale to agent i. The authors are able to explicitly construct the optimal mechanism under two particular information structures, assuming full commitment of the principal to the proposed mechaism. Jullien (2000)[9] analyzes a general asymmetric information model with one principal and one agent, where the agent's outside option depends on his type, and characterizes the optimality conditions for the principal under full participation as well as optimal exclusion. In Figueroa and Skreta (2008)[6], the above two formulations are combined: they set up a model where externalities can take a general form, and at the same time the agents' outside options depend on other agents' decisions and their own private information. Unfortunately, they can only characterize the optimal mechanism when the agents' outside option takes a very special form, and it is not so clear how to implement this mechanism with a feasible contract in reality. Aseff and Chade (2008)[1] analyze a problem of multi-unit auction and identity-dependent externalities among bidders, derive the seller's optimal mechanism and characterize its properties. In their model, the primary objective of the principal is to maximize monetary transfers from sale.

In terms of applications, first there is a rich class of literature on potential externalities in merger and acquisition, which dates back to Kamien and Zang (1990)[10], who model acquisition between firms in Cournot competition as a sequential game. Segal (1999[13], 2003[14]) has covered quite a few fundamental models on firm takeover and monopoly acquisitions in his renowned papers, and Rasmusen et al. (1991)[12] and Segal and Whinston (2000)[15] show in detail how a monopoly can take advantage of externalities among consumers to deter potential rivals from entering the market; other applications in industrial organization include discussions on horizontal mergers in an oligopolistic market (Farrell and Shapiro, 1990)[5] and optimal tender offer strategies for takeover (Burkart et al., 1998[3]; Burkart and Lee, 2010[4]). The idea of exploiting externalities and private information in contracting also provides interesting research topics in law, such as incentive problems among a group of plaintiffs that prevent a settlement (Stremitzer, 2010)[16] and buying the right for a harmful project (Guttel and Leshem, 2011)[7].

Compared to existing literature to the best of my knowledge, the model I present in the following has two features. First, the contract form is both explicit and simple, which makes it easy to construct concrete examples in illustration of theoretical properties. Secondly, the equilibrium notion is relatively strong, implying a strong implematation in the sense of mechanism design. In fact, as will be shown in Section 4, implementation with strictly dominant strategy emerges with the optimal contingent contract.

3 Basic Model: Single-Price Contract

Consider a risk-neutral principal (designated as "she") who is negotiating with a group of N agents (designated as "he") for a buyout project. Let $I = \{1, ..., i, ..., N\}$ denote the set of agents. Each agent holds one homogeneous share of some property; the principal's valuation on the whole property is an increasing function $v(n)$, where n denotes the number of shares acquired from the agents and $v(0) = 0$. Typically, $v(n)$ is increasing in n. Different functional forms of $v(n)$ can be assumed for different environments; in the most extreme case, $v(n) = v$ if $n = N$, 0 otherwise, which reflects the principal's need to buy out all the agents for realization of her valuation.

For any agent i, his type is denoted by θ_i, which is his private information. The types of agents are independently and identically distributed: $\theta_i \overset{iid}{\sim} F[0, \overline{\theta}]$, where F is continuously differentiable, and $f(\theta) = F'(\theta) > 0 \ \forall \theta$. I assume that F is common knowledge to both the principal and the agents.

Let $s_i = 0$ or 1 denote agent i's decision, where $s_i = 1$ represents selling and 0 represents keeping the share to himself. The utility of an agent of type θ, if he does not sell the share to the principal, is denoted by $u(\theta; N-n)$, where n is defined as above. I assume that this utility is increasing in θ, and agents impose positive externality on others by not selling. To be precise, $u(\theta; N-n)$ is continuously differentiable and strictly increasing in θ and strictly decreaing in n. Again, the functional form of u is common knowledge among the parties.

Consider the following one-shot game: the principal posts a single buyout price, t, and the agents decide simultaneously whether to accept ($s = 1$), reject ($s = 0$), or mix between the two.

3.1 Equilibrium among Agents

The equilibrium notion I adopt in this paper is Baysian Nash equilibrium. First, it is helpful to show that any Baysian Nash equilibrium is characterized by a single cutoff θ^*, where an agent sells his share to the principal if and only if his type is below θ^*.

Lemma 1. *In any Baysian Nash equilibrium, there must be a single cutoff θ^*, and each agent's equilibrium strategy s_i^* is given by*

$$s_i^*(\theta) = \begin{cases} 1, \ if \ \theta \leq \theta^* \\ 0, \ otherwise \end{cases}$$

Proof. Let $\widehat{u}_i(\theta; I/\{i\}) = \mathbb{E}[u(\theta; N-n)|s_{-i}^*]$ denote agent i's expected utility if he rejects the offer, where $I/\{i\}$ denotes the set of the agents other than i. It is first clear that $\widehat{u}_i(\theta; I)$ is strictly increasing in θ. It implies that in equilibrium, $\exists$ unique $\{\theta_i^*\}$ such that $\forall i$,

$$s_i^*(\theta) = \begin{cases} 1, \ if \ \theta \leq \theta_i^* \\ 0, \ otherwise \end{cases}$$

Now suppose for some i, j, $\theta_i^* < \theta_j^*$. Let $\widehat{u}'_i(\theta; I/\{i\}) = \mathbb{E}[u(\theta; n+1)|s_{-i}^*]$, note that

$$\widehat{u}_i(\theta; I/\{i\}) = F(\theta_j^*)\widehat{u}_i(\theta; I/\{i,j\}) + (1 - F(\theta_j^*))\widehat{u}'_i(\theta; I/\{i,j\})$$
$$\widehat{u}_j(\theta; I/\{j\}) = F(\theta_i^*)\widehat{u}_j(\theta; I/\{i,j\}) + (1 - F(\theta_i^*))\widehat{u}'_j(\theta; I/\{i,j\})$$

Since by assumption $\theta_i^* < \theta_j^*$, $\widehat{u}_i(\theta; I/\{i\}) < \widehat{u}_j(\theta; I/\{j\})$ $\forall \theta$. However, from the equilibrium conditions we also have $\widehat{u}_i(\theta_i^*; I/\{i\}) \geq t$ and $\widehat{u}_j(\theta_j^*; I/\{j\}) \leq t$, which implies $\widehat{u}_i(\theta_i^*; I/\{i\}) \geq \widehat{u}_j(\theta_j^*; I/\{j\})$, which is a contradiction. Therefore, $\theta_i^* = \theta^*$ $\forall i$. □

Let $U(\theta; \theta^*, N)$ denote the expected payoff of a type θ agent from rejection, then

$$U(\theta; \theta^*, N) = \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!}(F(\theta^*))^n(1 - F(\theta^*))^{N-1-n}u(\theta; N-n)$$

where the distribution of n is $B(N-1, F(\theta^*))$. Note that $U(\theta;\theta^*,N)$ is strictly decreasing in θ^* since $B(N-1,(F(\theta'))$ strictly first-order stochastically dominates $B(N-1,F(\theta))$ $\forall\theta < \theta'$, and $u(\theta;N-n)$ is strictly decreasing in n. Correspondingly, let $V(\theta^*;N)$ denote the principal's expected value that can be realized from acquisition of the shares of property, we have

$$V(\theta^*;N) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} F(\theta^*)^n (1-F(\theta^*))^{N-1-n} v(n)$$

Let $g(\theta;N) = U(\theta;\theta,N)$. As can be easily seen, given a price t, there can be multiple equilibria as long as the solution to $U(\theta;\theta,N) = t$ is not unique, as shown in Figure 1 below.

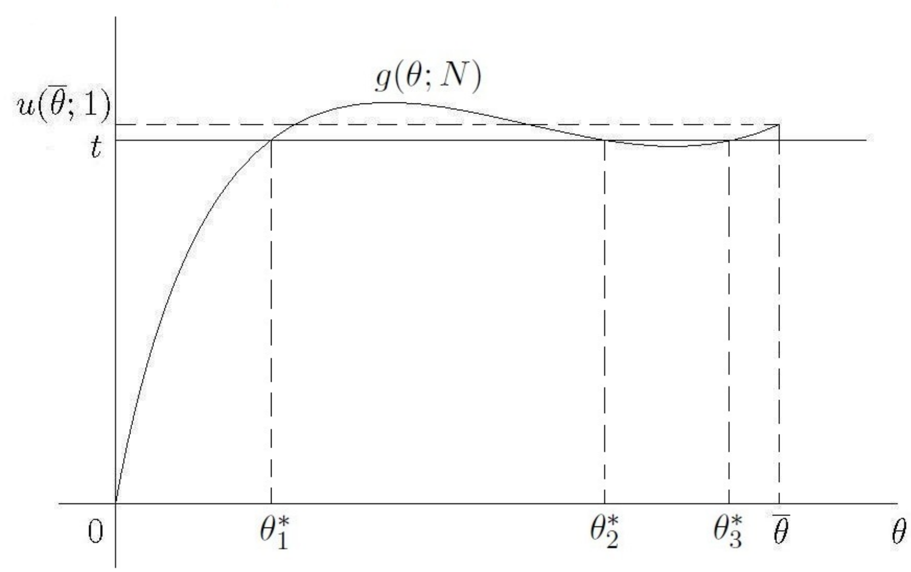

Figure 1: Multiple Equilibrium Cutoffs

Thus to provide a systematic analysis, we need to impose an equilibrium selection criterion. I hereby define an *equilibrium* to be one where selling is the only rationalizable strategy if and only if the agent's type is below the cutoff. On one hand, this definition is consistent with literature in empirical studies that have shown that individuals are reluctant to change their status quo; on the other hand, it can be regarded as the principal taking a most prudent point of view, taking rejection as a more likely action for the agents when there can be multiple equilibria.

As it turns out, the thus defined equilibrium indeed coincides with the lowest possible cutoff. In other words, the above refinement selects a unique equilibrium given any $t > u(0, N)$.

Theorem 1. *Assume that $u(0; N) < u(\overline{\theta}; 1)$. For any $t > u(0; N)$, if $\exists\, \theta \in [0, \overline{\theta}]$ such that $g(\theta; N) = t$, then the unique equilibrium among agents is given by*

$$s_i^*(\theta) = \begin{cases} 1, \; if \;\; \theta \leq \theta^* \\ 0, \; otherwise \end{cases}$$

where

$$\theta^* = \min\{\theta : g(\theta; N) = t\}$$

Proof. Let $\theta_1 = u^{-1}(t; N)$. I prove the result by construction:

1. By definition of θ_1, we first know that it is a strictly dominant strategy for $\theta \in [0, \theta_1)$ to sell. Then, note that $t = U(\theta_1; 0, N) > U(\theta_1; \theta_1, N) = g(\theta_1)$. Since $U(\theta; \theta^*, N)$ is strictly increasing in θ and strictly decreasing in θ^*, we know that $g(\theta; N) < t \;\forall \theta \in [0, \theta_1]$, thus no $\theta \in [0, \theta_1]$ is a solution to $g(\theta; N) = t$.

2. $\exists \theta_2 > \theta_1$ such that $U(\theta_2; \theta_1, N) = t$, otherwise there would be no solution to $g(\theta; N) = t$. Then after the first round of IESDS, it is a strictly dominant strategy for $\theta \in [0, \theta_2)$ to sell. Moreover, following a similar argument to that in step 1, no $\theta \in [0, \theta_2]$ is a solution to $g(\theta; N) = t$.

...

n. $\exists \theta_n > \theta_{n-1}$ such that $U(\theta_n; \theta_{n-1}, N) = t$. Then after the $(n-1)th$ round of IESDS, it is a strictly dominant strategy for $\theta \in [0, \theta_n)$ to sell. Moreover, following a similar argument to that in step 1, no $\theta \in [0, \theta_n]$ is a solution to $g(\theta; N) = t$.

...

By the above steps I construct a sequence $\{\theta_n\}$. Since this sequence is increasing and bounded by $\overline{\theta}$, it has limit θ^*. At θ^*, it is clear that $g(\theta^*; N) \leq t$ by construction, and

$g(\theta^*; N) \geq t$ since g is continuous, which imply that $g(\theta^*; N) = t$. Since by construction no $\theta \in [0, \theta^*)$ is a solution to $g(\theta; N) = t$, $\theta^* = \min\{\theta : g(\theta; N) = t\}$. Also by construction we know that s^* as specified above is the unique equilibrium, where uniqueness is given by the uniqueness of the limit. $\square$

Corollary 1. $\forall t > \max_{\theta'} g(\theta'; N)$, *the unique equilibrium is* $s(\theta) = 1\ \forall \theta$.

Proof. If $\nexists \theta_1$ such that $u(\theta_1; N) = t$, then it is a dominant strategy for any agent to sell. If there exists such a θ_1, then if $\nexists \theta_2$ such that $U(\theta_2; \theta_1, N) = t$, it would be a dominant strategy for any agent to sell, after the first round of IESDS. In this way we can pin down θ_1, θ_2,..., but for some finite n, θ_n cannot exist, because otherwise there would be a solution to $g(\theta; N) = t$, which is equal to $\lim_{n \to \infty} \theta_n$. That violates the assumption that $t > \max_{\theta'} g(\theta')$. Therefore, after finite rounds of IESDS, it would be a dominant strategy for any agent to sell. $\square$

3.2 Profit Maximization for Principal

Now, suppose we only consider the range of t such that $\{\theta : g(\theta; N) = t\} \neq \varnothing$. Define $h(t) = \min\{\theta : g(\theta; N) = t\}$, then the principal's problem is

$$\max_t V(h(t); N) - NF(h(t))t$$

Consider a range of t, $[t_1, t_2]$, where $h(t)$ is continuous. It implies that $h(t)$ is bijective between $[t_1, t_2]$ and $[h(t_1), h(t_2)]$, the latter of which I denote by $[\theta_1, \theta_2]$. Then the problem on $[t_1, t_2]$ becomes

$$\max_{\theta \in [\theta_1, \theta_2]} V(\theta; N) - NF(\theta)g(\theta; N)$$

First it is quite clear that the optimal contract exists in this constrained problem, since it is basically finding the maximum of a continuous function on a compact set. Then, even though $g(\theta; N)$ can take a quite irregular shape, as long as we can transform the choice set of t into the choice set of θ, which is a finite union of disjoint compact intervals, the optimal contract must exist.

In particular, we may be interested in the conditions under which the principal offers a high enough price to buy out every agent, regardless of their type. I first specify the set of regularity conditions, which I denote as condition R, and then show that it is sufficient for an optimal contract with a sufficiently high price.

Definition 1. *Let $\pi(\theta) = V(\theta; N) - NF(\theta)g(\theta; N)$. The model is said to satisfy condition R if*

(1) $\pi(\theta) > 0$ for some θ;

(2) $g(\theta; N)$ is strictly concave;

(3) $v(n)$ is convex: $\frac{v(n)}{n}$ is increasing in n.

(4) $\frac{f(\theta)}{F(\theta)}g(\theta; N)$ is decreasing wherever $g(\theta; N)$ is increasing;

(5) $\frac{f(\theta)}{F(\theta)}(\widehat{V}(\theta; N-1) - V(\theta; N-1))$ is increasing wherever $g(\theta; N)$ is increasing, with $\widehat{v}(n) = v(n+1)$.

Conditions (1)-(2) do not have much implications. (3) indicates that the marginal value of acquiring more shares for the principal is increasing in the number of shares already acquired. (4) can be treated as requiring the distribution of types to be sensitive enough to an increment in t: when $\frac{f(\theta)}{F(\theta)}$ decreases at a relatively higher rate whenever $g(\theta; N)$ increases, by raising t the principal would be able to capture a relatively larger proportion of the type of agents to accept the contract. On the other hand, (5) is a condition on the principal's revenue structure, which can be interpreted as $v(n)$ being "convex enough" in the sense that expanding the range of types to sell in equilibrium brings about a significant increment in the expected realized value. A straight forward example of an environment satisfying condition R is $\theta \sim iidU[0,1]$, $u(\theta; N-n) = (N-n)\theta$, $v(n) = v > 0$ if $n = N$ and 0 otherwise, and finally $N \geq 3$.

Theorem 2. *Let $\widetilde{\theta} = \arg\max_\theta g(\theta; N)$.*

a. For the principal, the minimum total payment to buy out all the agents regardless of their type is $Ng(\widetilde{\theta}; N)$.

b. Assume condition R. The unique optimal contract for the principal is $t = g(\widetilde{\theta}; N)$, and in equilibrium each agent sells with probability one, regardless of their type draws.

Proof. *a*. On one hand, if the principal sets t to be less than $g(\widetilde{\theta}; N)$, it is clear that the equilibrium cutoff θ^* is strictly less than $\overline{\theta}$, which implies that there are types of agents that would not sell to the principal in equilibrium. Thus the principal would need to set $t \geq g(\widetilde{\theta}; N)$ if she were to buy out the agents without any risk. On the other hand, we know from Corollary 1 that $t = g(\widetilde{\theta}; N)$ would indeed enable the principal to buy out every agent regardless of their type.

b. First, from (2) we know that $\widetilde{\theta}$ is unique. If the principal does not buy out each agent of every possible type, essentially she solves

$$\max_{\theta \in [0, \widetilde{\theta}]} \pi(\theta)$$

and we have

$$\begin{aligned}
\pi'(\theta) = & -N(1-F(\theta))^{N-1} f(\theta) v(0) \\
& + \sum_{n=1}^{N-1} \frac{N!}{n!(N-n)!} f(\theta)(nF(\theta)^{n-1}(1-F(\theta))^{N-n} - (N-n)F(\theta)^n(1-F(\theta))^{N-n-1})v(n) \\
& + NF(\theta)^{N-1} f(\theta) v(N) - N(F(\theta)g'(\theta; N) + f(\theta)g(\theta; N)) \\
= & \, NF(\theta)\left(\frac{f(\theta)}{F(\theta)}(\widehat{V}(\theta; N-1) - V(\theta; N-1)) - g'(\theta; N) - \frac{f(\theta)}{F(\theta)} g(\theta; N)\right)
\end{aligned}$$

By (2)(4)(5) we know that the expression $(\frac{f(\theta)}{F(\theta)}(\widehat{V}(\theta; N-1) - V(\theta; N-1)) - g'(\theta; N) - \frac{f(\theta)}{F(\theta)} g(\theta; N))$ is increasing in θ. Then as $\pi'(\theta) < 0$ for sufficiently small θ, by (1) we know that there is a unique θ' such that $\pi'(\theta') = 0$, where $\pi'(\theta) < 0$ for $\theta < \theta'$ and $\pi'(\theta) > 0$ for $\theta > \theta'$, and that $t = g(0; N) = u(0; N)$ is never optimal. Since now the optimal choice for the principal is a bang-bang solution, we can conclude that the unique optimal price is $t = g(\widetilde{\theta}; N)$. Finally, we compare the profit of this pricing scheme and one where the principal buys out everyone. From Corollary 1, the minimum price that the principal needs to pay to buy out each type of agent with probability one is also $t = g(\widetilde{\theta}; N)$ at the limit. And by (3),

$$
\begin{aligned}
v(N) - Nt &\geq F(\widetilde{\theta})v(N) - NF(\widetilde{\theta})t \\
&= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} F(\widetilde{\theta})^n (1 - F(\widetilde{\theta}))^{N-1-n} \frac{v(N)}{N} - NF(\widetilde{\theta})t \\
&> V(\widetilde{\theta}; N) - NF(\widetilde{\theta})t \\
&> 0
\end{aligned}
$$

the optimal contract is the one where each agent sells with probability 1, regardless of their type. $\square$

Below I provide two examples, as illustration of the optimal single-price contract. The first one satisfies condition R; in the second one, condition R is violated but the optimal contract can still be explicitly derived by the first order condition. However, now the principal may have to make "unnecessary"' payments in equilibrium. Without loss of generality, for all the examples in this paper I would use the following valuation structure for the principal:

$$
v(n) = \begin{cases} v, \ if \ n = N \\ 0, \ otherwise \end{cases}
$$

Example 1. *Assume that $\theta \in [0,1]$, $F(\theta) = \theta$, $u(\theta; N-n) = (N-n)\theta$, $N \geq 3$. Then*

$$
\begin{aligned}
g(\theta; N) &= \theta((N-1)(1-\theta) + 1) \\
&= N\theta - (N-1)\theta^2 \\
\widetilde{\theta} &= \frac{N}{2N-1}
\end{aligned}
$$

The optimal single-price contract is characterized by

$$
t^* = \frac{N^2}{4(N-1)}
$$

and her maximized profit is

$$
\pi^* = v - \frac{N^3}{4(N-1)}
$$

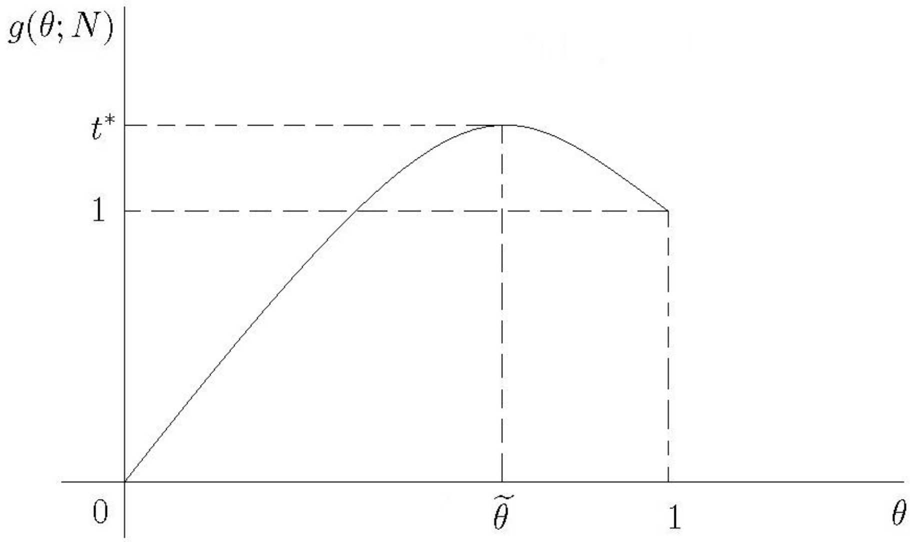

Figure 2: Illustration for Example 1

Example 2. *Assume that $\theta \in [0,1]$, $F(\theta) = \theta$, $u(\theta; N-n) = (N-n)\theta^2$, $N = 2$, $v = 2$. Then*

$$g(\theta; N) = 2\theta^2 - \theta^3$$

which is increasing on $[0,1]$. And

$$\begin{aligned} R'(\theta) &= 2\theta(4\theta^2 - 6\theta + 2) \\ &= 2\theta(4\theta - 2)(\theta - 1) \end{aligned}$$

Therefore if the principal does not buy out every type of agent, her optimal strategy is to set $t = g(\frac{1}{2}) = \frac{3}{8}$, and her expected profit is $\frac{1}{8}$. If she buys out every type of agent, then $t = g(1) = 1$, and her profit is 0. Therefore the optimal contract is $t = \frac{3}{8}$ and the probability of the principal making an "unnecessary" payment is $\frac{1}{2}$.

4 Contingent Contract

Based on the results for the single-price contract, now we can take one step further and allow the principal to propose a contingent contract to each agent. By symmetry of the problem, this contract is first anonymous, i.e. the contract is independent of the identities of agents; it's also a take-it-or-leave-it offer without further complication. The difference between this generalized contract and the one described in the previous section is that, the principal can now condition the payment to any agent who accepts on the number of accepting agent (again since the agents are homogeneous ex ante, WOLOG we don't have to consider the agents' identities).

Formally, the contract terms include a series of possible payments $\{t_n\}$, $n = 0, 1, ..., N-1$ where the subscript denotes the number of agents, not including the agent being offered, who turn out to accept their contracts. For instance, t_2 denotes the payment that the agent would get if there are two other agents accepting their offers, so the total number of accepting agents would be 3.

4.1 Equilibrium among Agents

I adopt the same equilibrium concept as defined in the last section. To reiterate, the principal is taking the most conservative view and is assuming the "worst" equilibrium to emerge. Given a cutoff rule θ^*, which means that an agent would accept the contract and sell his share of property to the principal if and only if his type $\theta \leq \theta^*$, define

$$\begin{aligned} t(\theta^*; N) &= \mathbb{E}[t_n|\theta^*] \\ &= \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!} F(\theta^*)^n (1 - F(\theta^*))^{N-1-n} t_n \end{aligned}$$

Essentially, $t(\theta^*; N)$ denotes the agent's expected payoff if he accepts the contract, and the equilibrium is characterized by the equation

$$g(\theta^*; N) = t(\theta^*; N)$$

by the argument provided in the last section.

As it turns out, there does not exist a parallel result to Theorem 1 in the case of contingent contract, the reason being that the technique of constructing a sequence $\{\theta_n\}$ is no longer valid, due to the possibility that $t(\theta; N)$ takes different values on different θ's. However, as the following result shows, there is a particular type of contingent contract, which enables the principal to set any $\theta^* \in [0, \overline{\theta}]$ as the equilibrium cutoff. Moreover, the resulting equilibrium is even stronger than previously defined: under this specific contingent contract we obtain a dominant strategy equilibrium.

Theorem 3. *Fix $\theta^* \in [0, \overline{\theta}]$. Let $t_n = u(\theta^*; N-n)$, then*

a. θ^* is the unique Baysian Nash equilibrium cutoff.

b. In equilibrium, accepting is a strictly dominant strategy for type $\theta < \theta^*$, and rejection is a strictly dominant strategy for type $\theta > \theta^*$.

Proof. *a.* Note that first by definition of $t(\theta; N)$ and $g(\theta; N)$, we know that $g(\theta^*; N) = t(\theta^*; N)$, and thus θ^* is indeed an equilibrium cutoff in this payment scheme. To show uniqueness, simply observe that

$$t(\theta; N) - g(\theta; N) = \sum_{n=0}^{N-1} \frac{(N-1)!}{n!(N-1-n)!} F(\theta^*)^n (1 - F(\theta^*))^{N-1-n} (u(\theta^*; N-n) - u(\theta; N-n))$$

The above expression is strictly positive if and only if $\theta < \theta^*$, and strictly negative if and only if $\theta > \theta^*$. Therefore, there can be only one solution to the equation $g(\theta; N) = t(\theta; N)$.

b. For any agent of type θ, given that n other agents accept the contract, if he accepts the contract he gets $t_n = u(\theta^*; N-n)$, while if he rejects he gets $u(\theta; N-n)$. It is then clear that $\forall \theta < \theta^*$, it is a dominant strategy for the agent to accept, while $\forall \theta > \theta^*$ it is a dominant strategy for the agent to reject. □

In fact, the above proof applies to an arbitrary continuous joint distribution function $F(\theta_1, ..., \theta_N)$, where the agents' types may be correlated.

Corollary 2. *Given any continuous joint distribution function $F(\theta_1, ..., \theta_N)$, $\theta_i \in [0, \overline{\theta}]$ $\forall i$, the principal can implement any $\theta^* \in [0, \overline{\theta}]$ as the unique equilibrium cutoff by a strictly dominant strategy Nash equilibrium.*

The contingent contract as specified above brings about two advantages. First, we now have only one intersection of $g(\theta; N)$ and $t(\theta; N)$, which represents a dominant strategy Baysian Nash equilibrium, i.e. we get full implementation in this case. Moreover, since now $t(\theta; N)$ can be a non-constant function (as shown in the figure below), we do not have to limit our attention to the increasing part of $g(\theta; N)$ when we consider the principal's profit maximization problem, which implies intuitively that the principal is better off with the option of writing a contingent contract. In the next subsection I formalize this argument.

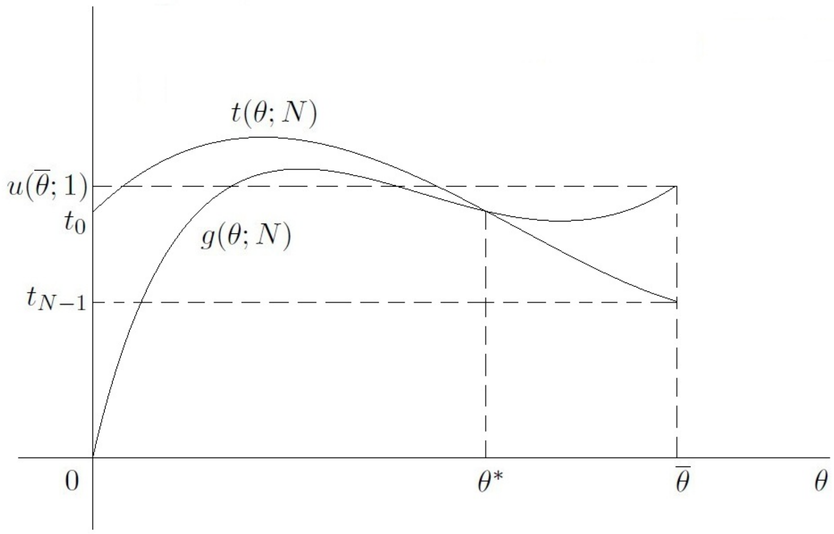

Figure 3: Non-constant Expected Payment in Contingent Contract

4.2 Profit Maximization for Principal

With the option of contingent contract, the principal is less constrained for profit maximization, thus it is conceivable that the principal is probably going to set $t(\theta; N)$ differently from the restricted case of single-price contract.. As the next result shows, the principal does select a different equilibrium in the optimal contingent contract.

Theorem 4. *a. The principal needs to pay at least* $Nu(\overline{\theta}; 1)$ *to buy out all the agents, regardless of their type.*

b. Assume condition R, then the optimal contract for principal is

$$t_n = u(\overline{\theta}; N - n)$$

The induced equilibrium is that each agent sells to the principal, regardless of their type.

Proof. *a*. To buy out each agent regardless of type, the principal pays each agent t_{N-1} in equilibrium. Since an agent would get $u(\theta;1)$ if he rejects, we know that $t_{N-1} \geq u(\overline{\theta};1)$. Note that by Theorem 3, it can be achieved that $\{\theta \in [0,\overline{\theta}) : t(\theta;N) = g(\theta;N)\} = \varnothing$, thus there indeed exists an equilibrium among agents such that any type of agent sells to the principal for payment $t_{N-1} = u(\overline{\theta};1)$.

b. Suppose the principal does not buy out each agent for sure regardless of their type. The principal solves

$$\max_{\theta \in [0,\overline{\theta}]} V(\theta;N) - Ng(\theta)F(\theta)$$

From Theorem 2, we know that $\widetilde{\theta} = \arg\max g(\theta)$ dominates any $\theta \in [0,\widetilde{\theta})$, thus the problem essentially becomes

$$\max_{\theta \in [\widetilde{\theta},\overline{\theta}]} V(\theta;N) - Ng(\theta)F(\theta)$$

But then from previous analysis we also know that

$$v(N) - Ng(\theta;N) > V(\theta;N) - Ng(\theta;N)F(\theta)$$

if

$$V(\theta;N) - Ng(\theta;N)F(\theta) > 0$$

Thus the problem for principal can be further relaxed into

$$\max_{\theta \in [\widetilde{\theta},\overline{\theta}]} v(N) - Ng(\theta)$$

whose solution is $\overline{\theta}$ given condition R. Finally, by Theorem 3, $\overline{\theta}$ is an achievable solution to the relaxed problem, which implies that the optimal contract is the one specified above. $\square$

Now we may as well take a second look at Example 1. The principal is strictly better off with the option of a contingent contract.

Example 3. *Assume the same parameter values as in Example 1. Allowing for contingent contracts, at optimum the principal sets*

$$t_n = (N - n)$$

while in equilibrium she pays 1 *to each agent. Thus her maximized profit is*

$$\pi^* = v - N$$

which is strictly higher than $v - \frac{N^3}{4(N-1)}$, *the maximized profit under the optimal single-price contract, if* $N \geq 3$.

An illustration of comparison between the equilibrium under the optimal contract for principal is given below. The left hand side shows the single-price contract while the right hand side shows the contingent contract. As can be easily seen from the graph, the contingent contract dominates the single-price one from the principal's point of view; in fact, under condition R, the principal is strictly better off with the option of writing a contingent contract.

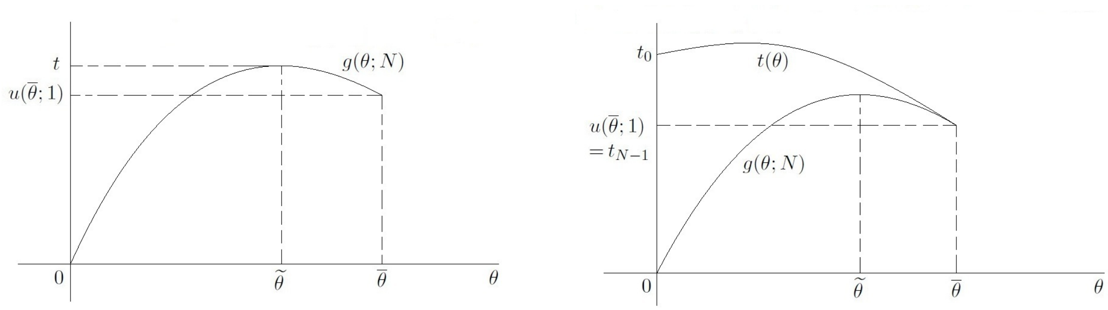

Figure 4: Comparison between Two Contracts

Corollary 3. *Assume condition R, then the principal's payoff is strictly higher under the optimal contingent contract than the single-price one.*

In general, without imposing much restriction on g, we can derive a sufficient and necessary condition for the strict dominance of contingent contract over single-price contract, as long as $\{t_n\}$ can be chosen freely:

Corollary 4. *Let* π^* *denote the principal's expected payoff under the optimal single-price contract. The principal's expected payoff is strictly higher under the optimal contingent contract than the single-price one if and only if* $\exists\, \theta'$ *such that* $V(\theta'; N) - NF(\theta')g(\theta'; N) > \pi^*$.

The previous analysis implies that, though simple in contract terms, the above contingent contract is rather powerful in the sense that it enables the principal to select any θ^* as an equilibrium payoff, and implement it with a strictly dominant strategy equilibrium. It is then reasonable to conceive that this particular contract may be optimal within some class of general mechanisms. The following section formally proves this conjecture.

5 General Commitment Scheme

Consider the following mechanism Γ:

Step 1. The principal proposes a payment scheme $\{t_i(\{m_j\}_{j=1}^N)\}_{i=1}^N$, where t_i denotes the payment to agent i and m_j denotes the message sent by agent j. The message space, M[1], must satisfy $R \in M$ where R denotes the option of rejecting the principal's proposal; the payment function must satisfy $t_i(R, m_{-i}) \geq 0$ by the individual rationality for agent i. In order to identify the optimal mechanism for the principal, it is then without loss of generality to focus our attention only on the class of mechanisms where $t_i(R, m_{-i}) = 0 \ \forall i$.

Step 2. For any agent i, he chooses a message $m_i \in M$ to report. He sells his share of property to the principal for $t_i(\{m_j\}_{j=1}^N)$ if and only if $m_i \neq R$.

To be coherent with the previous analysis, the equilibrium notion I adopt here is the Baysian Nash equilibrium. Given $\{t_i(\{m_j\}_{j=1}^N)\}_{i=1}^N$, an *equilibrium* is a mapping $m^*(\theta) : [0, \overline{\theta}] \longrightarrow M$ such that

$$m^*(\theta) = \begin{cases} \arg\max_{m \neq R} \mathbb{E}[t_i(m, m^*(\theta_{-i}))], \ \ if \ \ \max_{m \neq R} \mathbb{E}[t_i(m, m^*(\theta_{-i}))] \geq \mathbb{E}[u(\theta; N-n) | m^*] \\ R, \ otherwise \end{cases}$$

First we can show the following by a standard Revelation Principle argument:

Lemma 2. *Any equilibrium in Γ can be implemented as an equilibrium in a direct revelation mechanism Γ' satisfying individual rationality and incentive compatibility, where $M = [0, \overline{\theta}] \cup \{R\}$ and $m^*(\theta) = \theta$ if $m^* \neq R$.*

[1]Due to the ex ante homogeneity of agents by assumption, I only consider the symmetric case. The same principle applies to the equilibrium notion defined below.

With the above lemma, we can constrain the analysis within the scope of incentive compatible direct revelation mechanisms, denoted by Γ'. Individual rationallity is trivially satisfied in any Γ' by the inclusion of R in M, thus without loss of generality I supress the notion of individual rationality in the analysis below. From incentive compatibility we obtain a second lemma:

Lemma 3. *$\forall \Gamma'$, if $m^*(\theta') = \theta'$ for some θ', then*

(1) $m^*(\theta) = \theta \ \forall \theta \leq \theta'$

(2) $\mathbb{E}[t_i(\theta, \theta_{-i})] = \mathbb{E}[t_i(\theta', \theta_{-i})] \ \forall \theta \leq \theta'$.

Proof. From $m^*(\theta') = \theta'$ we know that $\mathbb{E}[t_i(\theta', \theta_{-i})] \geq \mathbb{E}[u(\theta'; n)|m^*]$, which implies that $\mathbb{E}[t_i(\theta', \theta_{-i})] > \mathbb{E}[u(\theta; N-n)|m^*] \ \forall \theta \leq \theta'$. Note that for agent i of type $\theta < \theta'$, he could at least get $\mathbb{E}[t_i(\theta', \theta_{-i})]$ by reporting θ', thus R is strictly dominated, and (1) is proved. Now, suppose $\mathbb{E}[t_i(\theta, \theta_{-i})] > \mathbb{E}[t_i(\theta', \theta_{-i})]$ for some $\theta < \theta'$, then by choosing $m(\theta') = \theta$ the agent gets a strictly higher payoff, which violates incentive compatibility. A similar argument shows that $\mathbb{E}[t_i(\theta, \theta_{-i})] < \mathbb{E}[t_i(\theta', \theta_{-i})]$ would also violate incentive compatibility. Therefore, (2) holds as well. □

The next lemma, as a direct result of Lemma 5, enables us to characterize an equilibrium in Γ' as a cutoff θ^*:

Lemma 4. *For any equilibrium in Γ', there exists a cutoff θ^* such that*

$$m^*(\theta) = \begin{cases} \theta, \ \forall \theta \leq \theta^* \\ R, \ otherwise \end{cases}$$

Moreover, $\mathbb{E}[t_i(\theta^*, \theta_{-i})] = g(\theta^*; N)$.

Proof. The first claim follows directly from Lemma 5. For the second one, note that with $m^*(\theta)$ thus specified, $\mathbb{E}[u(\theta; N-n)|m^*] = U(\theta; \theta^*, N)$. Therefore

$$\mathbb{E}[t_i(\theta^*, \theta_{-i})] \geq U(\theta^*; \theta^*, N)$$
$$\mathbb{E}[t_i(\theta^*, \theta_{-i})] < U(\theta; \theta^*, N) \ \forall \theta > \theta^*$$

which then implies $\mathbb{E}[t_i(\theta^*, \theta_{-i})] = g(\theta^*; N)$ by the continuity of $U(\theta; \theta^*, N)$. □

Now we can show that among all general mechanisms Γ, the contingent contract described in section 4 is optimal for the principal.

Theorem 5. *Let $\mathbb{G}$ denote the set of all Γ's satisfying the criteria described before, and let Γ^* denote the contingent contract described in section 4. Let $R(\Gamma)$ denote the principal's expected payoff from mechanism Γ, then $R(\Gamma^*) \geq R(\Gamma)$ $\forall \Gamma \in \mathbb{G}$.*

Proof. Note that Γ^* is equivalent to a direct revelation mechanism where

$$t_i = \begin{cases} u(\theta^*, N-n), \ if \ m_i \neq R \\ 0, \ otherwise \end{cases}$$

where $n = \sharp\{i : t_i \neq R\}$. By Lemma 4, it suffices to prove the result within $\mathbb{G}' \subset \mathbb{G}$, the set of all incentive compatible direct revelation mechanisms. Then by Lemmas 5 and 6, we can write the principal's relaxed problem as

$$\max_{\theta \in [0,\overline{\theta}]} V(\theta; N) - NF(\theta)g(\theta; N)$$

This problem is relaxed because it is not proved yet that for each $\theta \in [0, \overline{\theta}]$ there exists $\Gamma' \in \mathbb{G}'$ such that θ is the equilibrium cutoff.

First, by continuity of the maximand and compactness of $[0, \overline{\theta}]$, the optimal θ exists in the relaxed problem. Secondly, by Theorem 3, we know that Γ^* enables the principal to choose any $\theta \in [0, \overline{\theta}]$ as the equilibrium cutoff. Therefore, the optimal cutoff in Γ^* solves the relaxed problem, which implies that Γ^* is indeed the optimal mechanism for the principal in $\mathbb{G}$. $\square$

Theorem 5 provides a somehow surprising assertion: as long as the information structure is independent and ex-ante homogeneous, the principal can fully implement any symmetric Baysian Nash equilibrium using a straight forward contingent contract as described in Section 4.

6 Conclusion

I have analyzed the optimal contract design problem faced by the principal, when she is negotiating with a group of agents in the presence of externalities and asymmetric information. In the model setting of this paper, the optimal contract exists in both contexts, and condition R characterizes a set of conditions sufficient for the principal to buy out each agent regardless of their types. In applications, this can be viewed as the characteristics of the information and payoff structure such that a hostile firm offers a substantially high takeover bid or tender offer premium, or a defendant offers a highly attractive amount of compensation in order for a settlement out of court.

Also, the intuition raised in the introduction section is confirmed. Due to externalities, the principal tends to make a considerably higher payment to an agent than when dealing with him independently. Indeed, if the agents hold their shares separately, each would get utility $u(\theta;1)$ as an outside option, thus to persuade the agent the principal needs to offer at most $u(\overline{\theta};1)$; but with externalities, in equilibrium the principal would in general has to pay a higher amount to the agents that accept, as in the case of single-price contract. Nevertheless, the optimal contingent contract attenuates this effect, as can be seen from the optimal contingent contract under condition R, where the principal pays exactly $u(\overline{\theta};1)$ to each agent. In terms of asymmetric information, if we compare the expected payment to the accepting agents and the status quo utility they would get had there not been this project, $u(\theta;N)$, it is clear that asymmetric information benefits agents with low θ and harms those with high θ. In fact, the effect of information asymmetry can be decomposed into two: on one hand, the principal has to make a sufficiently high offer as the outside option plus informational rent; on the other hand, the principal does not need to make the offer too high as long as it is sufficient to make agents believe that others probably would accept. The first effect is constant given the types of contract that the principal is allowed to choose from. As for the second, it forces all agents with type below the equilibrium cutoff to accept the offer in fear that others may accept anyway, but for higher types they have to give up more than they could receive in expectation.

One among a few limitations of this model is the assumption of the principal's full commitment power, as present in a wide range of mechanism design literature. After all, it is not always incentive compatible for the principal to commit to his proposed contract: for example, in the case that the principal has to acquire all the shares to realize her valuation, if any one of the agents rejects in the end, the principal would rather abandon the project without making unnecessary payments. To practically solve this problem, one possibility is to exogenously impose a sufficiently high cost of default, such as the requirement of notarizing the contract via a legal process. Another way to get over this lies in the ability of verification of the agents: either the contract is publicly observable, or the agents can costlessly verify the outcome. In the apartment building conversion project, for example, the agents can easily tell if the principal has paid according to the number of accepting agents by counting the number of empty apartments in the building afterwards. Another issue is that the assumption of a one-shot game may be over-simplifying. If the principal does not end up getting a satisfactory number of shares, it is easily conceivable that she would make an attempt to continue bargaining with the remaining agents. Therefore, application of this model lies primarily in strategic situations where it is rather costly to organize a negotiation process, or the buyout project has to be completed within a very limited period of time.

There are a few directions for future research that are worth pursuing. First, within the scope of this model, one can investigate the explicit form and properties of the optimal contract when the externalities are negative, as in the case of merger among firms in Cournot competition. The condition inducing the optimal contract to achieve exact or asymptotic efficiency is also a non-trivial issue. Secondly, as mentioned above, analysis under different protocols would extend the application of the basic idea. One may be inclined to consider (1) sequential bargaining with discount factor; (2) divide-and-conquer strategy by the principal; (3) contract offered by one or a group of agents; (4) the possibility of coordination among agents, etc. Last but not least, it is also interesting to analyze the problem assuming noisy observations, meaning that either the principal's offer and/or the agents' decision can only be observed with some noise, and see whether such perturbation creates additional incentive problems, whether it is benefitial for the principal, and whether there exists a convergence result.

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