ECON201B TA Section: Week 1

Yangbo Song

1 Extensive Form Game

· An extensive form game, $G = (N, H, P, f_c, \{g_i\}_{i=1}^{I})$, consists of

– a. A (finite) set of players: $N = \{1, ..., I\}$;

– b. A set of histories H such that (1)$\varnothing \in H$; (2)$(a^1, ..., a^k) \in H \to (a^1, ..., a^l) \in H$ $\forall l < k$; (3)$(a^1, ...) \in H$ if $(a^1, ..., a^k) \in H$ $\forall k = 1, 2, ...$;

– c. A set of terminal histories $Z \subset H$: $(a^1, ..., a^k) \in Z \Leftrightarrow \nexists a^{k+1}$ such that $(a^1, ..., a^{k+1}) \in H$;

– d. A function $P : H \setminus Z \to N \cup \{c\}$ that defines the player who moves at each history (c denotes nature);

– e. A function f_c that assigns a probability distribution $f_c(h) \in \Delta(A_c(h))$. $A_c(h)$ is the set of available actions by nature at history h: $A_c(h) = \{a : (h, a) \in H, P(h) = c\}$ ($A_i(h)$ below is defined similarly);

– f. A partition $\mathcal{I}_i$ of $H_i = \{h \in H : P(h) = i\}$ $\forall i \in N$ such that $A_i(h) = A_i(h')$ $\forall h, h' \in I_i \in \mathcal{I}_i$, $\forall i \in N$;

– g. A utility or payoff function for each player $i \in N$: $g_i : Z \to \mathbb{R}$.

· Agent i's *strategy* is defined in the following way:

– a. Define $A(I_i) = A_i(h) : h \in I_i$;

– b. A (pure) strategy for player i is a mapping s_i that assigns an action in $A(I_i)$ to each I_i;

– c. Example: Matching Pennies version A. One strategy for player 2 is $(h|H, h|T)$, namely h no matter what player 1's action is.

2 Normal Form Game

· A normal form game, $G = (N, \{S_i\}_{i=1}^I, \{g_i\}_{i=1}^I)$, consists of

– a. A (finite) set of players: $N = \{1, ..., I\}$;

– b. For each player i, a set of actions(strategies) $S_i = \{s_i\}$;

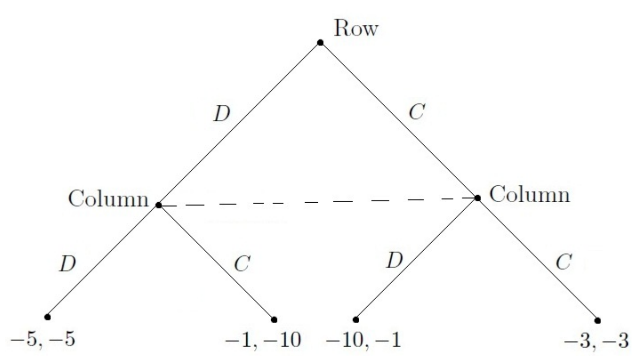
– c. A utility or payoff function for each player $i \in N$: $g_i : S_1 \times ... \times S_I \to \mathbb{R}$. Note that g_i here is different from that in an extensive form game: now g_i denotes the *expected* payoff for agent i.

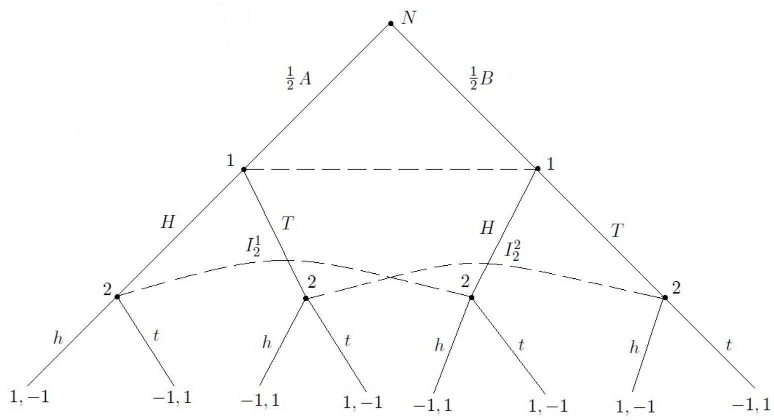
3 Transformation between the Two Game Forms: Examples

· From normal to extensive: consider the Prisoners' Dilemma. The first number in each cell is Row's payoff while the second number is Column's payoff.

		Column	
		D	C
Row	D	$-5, -5$	$-1, -10$
	C	$-10, -1$	$-3, -3$

We can transform it into an extensive form game by putting the two decision nodes for the player who moves second in one information set. Note that there is more than one way of transformation.





· From extensive to normal: consider the following variation of Matching Pennies. To transform it into a normal form game, first write out all strategies for each player, then identify the payoff vector for each strategy combination.

		2			
		$h\|I_2^1, h\|I_2^2$	$h\|I_2^1, t\|I_2^2$	$t\|I_2^1, h\|I_2^2$	$t\|I_2^1, t\|I_2^2$
1	H	$0, 0$	$0, 0$	$0, 0$	$0, 0$
	T	$0, 0$	$0, 0$	$0, 0$	$0, 0$

4 Exercise

4.1 Dominance Solvability

Consider the following normal form game:

		2		
		L	M	R
1	U	$4, 10$	$3, 0$	$1, 3$
	D	$0, 0$	$2, 10$	$10, 3$

Is it dominance solvable? If yes, show the steps of IESDS. If no, briefly state the reason.

Now consider the following normal form game, which is a slight generalization of the above:

		2		
		L	M	R
1	U	$4, 10$	$3, 0$	$1, 3$
	D	$0, 0$	$2, 10$	$10, x$

What is the condition on x which would make the game dominance solvable?

Solution. *In the original game, consider a mixed strategy* $\frac{1}{2}L + \frac{1}{2}M$. *For player 2, no matter what strategy player 1 chooses, the mixed strategy would yield him a payoff of* $5 > 3$, *thus* R *can be eliminated. Then it is clear that* D *can be eliminated for player 1 and then* M *for player 2, and the game is dominance solvable.*

In the second game, first it can be observed that U, D, L *cannot be eliminated for the corresponding players in the first round of IESDS (iterated elimination of strictly dominated strategies). For* M *to be eliminated, we must have* $x > 10$, *but after eliminating* M *no strategy can be eliminated in the next round, which is a contradiction. Now given that* $x \leq 10$, *to eliminate* R *we must construct a mixed strategy* $pL + (1-p)M$ *such that* $10p > 3$ *and* $10(1-p) > x$. *For there to be a* p *satisfying these two conditions, we have* $x \in (-\infty, 7)$.

4.2 Second-price Auction

A second-price auction is a sealed-bid auction where the highest bidder gets the object and pays the second highest bid. If there are more than one bidder with the highest bid, then the winner is decided by a equal-probability random draw. Assume that valuations are private (that is, each bidder's valuation for the object does not affect any other bidder's valuation) and that valuations of bidders are iid ex ante. Show that it is a weakly dominant strategy for each bidder to bid his own valuation.

Solution. *Consider bidder* i. *Let* b' *be the highest bid among bidders other than* i *and* v_i *be the valuation for bidder* i. *If* $v_i \geq b'$, *then it is weakly best to bid* v_i *in order to win the object and get* $v_i - b' \geq 0$; *if* $v_i < b'$, *then it is also weakly best to bid* v_i *to lose and get 0, rather than* $v_i - b' < 0$.

4.3 Median Voter Theorem

Consider an election with two candidates A and B. There are a continuum of citizens whose most preferred policies are distributed continuously on $[0, 1]$ with cdf F. The candidates choose their policy simutaneously from $[0, 1]$. Each citizen then votes for the candidate whose policy is closest to their most preferred one. The candidate with the majority of votes wins. What would be the weakly dominant strategy for a candidate?

Solution. *The weakly dominant strategy is to choose x such that $F(x) = 1 - F(x)$. Let y denote the opponent's strategy, if $y \neq x$, then x guarantees a sure win; if $y = x$, then x yields a $\frac{1}{2}$ chance of winning while any other strategy would lead to a sure loss.*

4.4 Public Good Provision

There are n villagers who have to decide whether to build a bridge. The value of the bridge to villager i is $v_i \in [0, \overline{v}]$, which is private information to i. At the beginning of the bridge project, each v_i is chosen by nature independently from some common distribution F. The total cost of building the bridge is C. Consider the following mechanism: each villager reports his or her valuation. If the sum of valuation is equal or greater than C, then the bridge is built and each villager is taxed $\frac{C}{n}$; otherwise the bridge is not built.

a. Name one strategy for a villager.

b. What is the weakly dominant strategy for each villager?

c. Suppose that each villager follows the weakly dominant strategy. Is the outcome always Pareto efficient?

Solution. *a. Report $v_i' = \frac{1}{2} v_i$.*

b. To report $\overline{v}$ if $v_i \geq \frac{c}{n}$, and 0 otherwise.

c. No. An easy example would be that $n = 2$, $v_1 = \frac{2}{3}\overline{v}$, $v_2 = 0$ and $C = \frac{5}{6}\overline{v}$.

ECON201B TA Section: Week 2

Yangbo Song

1 Equivalence between NBR and Strictly Dominated Strategy

Theorem 1. *For a finite strategic game, $s_i \in S_i$ is never-best response if and only if it is strictly dominated.*

Before proving the above theorem (somewhat informally), it is crucial to notice that correlated beliefs have to be allowed for this equivalence. In particular, the set of correlated beliefs is denoted $\Delta(S_{-i})$, where $S_{-i} = \times_{j \neq i} S_j$ is the set of the pure strategy profile for players other than i, as opposed to the set of independent beliefs $\times_{j \neq i} \Delta(S_j)$.

The proof of the theorem is as follows. First I define a $zero-sum\ game$ as a strategic game where the players' payoffs always sum up to 0. The following lemma is stated without proof:

Lemma 1. *(Minimax Theorem) In a two-player zero-sum game, there exists a strategy profile $(\alpha_1^*, \alpha_2^*) \in \Delta(S_1) \times \Delta(S_2)$ such that*

$$v_i(\alpha_1^*, \alpha_2^*) := \mathbb{E}[g_i(s_1, s_2)|\alpha_1^*, \alpha_2^*] = \max_{\alpha_1} \min_{\alpha_2} v_i(\alpha_1, \alpha_2) = \min_{\alpha_2} \max_{\alpha_1} v_i(\alpha_1, \alpha_2)$$

$\forall i = 1, 2.$

The "if" part of the theorem is trivial. For the "only if" part, suppose that s_i' is NBR. Consider tht following auxiliary zero-sum game with two hypothetical players 1 and 2:

1. Player 1's set of pure strategies is $S_i \setminus s_i'$, player 2's set of pure strategies is S_{-i}.

2. Player 1's payoff is $u_1(\alpha_1, \alpha_2) = v_i(\alpha_1, \alpha_2) - v_i(s_i', \alpha_2)$, player 2's payoff is $u_2 = -u_1$.

Now since s_i' is NBR, $\min_{\alpha_2} \max_{\alpha_1} u_1(\alpha_1, \alpha_2) > 0$. Then by Lemma 1, $\max_{\alpha_1} \min_{\alpha_2} u_1(\alpha_1, \alpha_2) > 0$. This means that there exists $\alpha_i^* \in \Delta(S_i \setminus s_i')$ such that $v_i(\alpha_i^*, s_{-i}) > v_i(s_i', s_{-i})$ $\forall s_{-i}$.

2 Exercise

2.1 Cournot Competition with 3 Players

Consider the following Cournot competition game: the inverse market demand is given by $P = a - Q$ $(a > 0)$ where Q denotes total production. There are 3 firms with the same constant marginal cost 0. Show that the game is not dominance solvable.

Solution. *It is easy to first derive that the best response for firm i to a total production by the other two firms q_{-i} is $\gamma(q_{-i}) = \frac{a-q_{-i}}{2}$ (the game is symmetric, so I omit the subscript for γ. Thus we can eliminate $(\frac{a}{2}, +\infty)$ for i in the first round of IESDS. But for the second round, since now the range of q_{-i} is $[0, a]$, no strategy can be eliminate for i. Thus the game is not dominance solvable.*

2.2 Role of Correlation in Rationalizability

Consider the following 3-player game: the pure strategies are U, D for player 1, L, R for player 2 and A, B, C for player 3. The payoffs are given in the following tables:

A

	L	R
U	$(1, 2, 1)$	$(2, 4, 10)$
D	$(2, 3, 10)$	$(0, 0, 3)$

B

	L	R
U	$(0, 5, 2)$	$(1, 6, 0)$
D	$(1, 2, 0)$	$(7, 2, 2)$

C

	L	R
U	$(0, 0, 3)$	$(3, 1, 10)$
D	$(1, 1, 10)$	$(4, 3, 1)$

Before deleting any strategy for players $1, 2$, is B strictly dominated? Is it NBR to independent randomization? Is it NBR to correlated randomization?

Solution. *B is not strictly dominated because no mixture of A and C can yield a strictly higher payoff in both cases (U, L) and (D, R). Denote mixed strategies for players 1 and 2 as $pU + (1-p)D$ and $qL + (1-q)R$, and consider a mixed strategy $s = aA + (1-a)C$*

for player 3. The difference in expected payoff for 3 between s and B is

$$
\begin{aligned}
&a(pq + 3(1-p)(1-q) + 10p(1-q) + 10q(1-p)) \\
&+ (1-a)(3pq + (1-p)(1-q) + 10p(1-q) + 10q(1-p)) - (2pq + 2(1-p)(1-q)) \\
=&(1-2a)pq + (2a-1)(1-p)(1-q) + k(p,q)
\end{aligned}
$$

where $k(p,q)$ is strictly positive for any $p, q \in [0,1]$. Note that for any $p, q \in [0,1]$, there always exists $a \in [0,1]$ such that $(1-2a)pq + (2a-1)(1-p)(1-q) \geq 0$. Therefore, B is NBR to independent randomization. However, it is not NBR to correlated randomization since it is BR to $\frac{1}{2}(U,L) + \frac{1}{2}(D,R)$.

ECON201B TA Section: Week 3

Yangbo Song

1 Sequential Rationality in Extensive Form Games of Perfect Information

In class we discussed backward induction, which yields a pure strategy NE. That particular NE is named a subgame perfect Nash equilibrium (SPNE). To define such an equilibrium explicitly, I first introduce the definition of a subgame.

Definition 1. *(Informal[1]) A subgame of an extensive form game of perfect information is one constructed by taking a node other than the terminal nodes as the initial node.*

Essentially, backward induction is the process of choosing best responses for each player in each subgame. Thus the concept SPNE is defined as follows:

Definition 2. *A strategy profile in an extensive form game is a SPNE if it induces a NE in every subgame.*

Proposition 1. *Every finite extensive form game of perfect information has a pure strategy SPNE. Moreover, if no player has the same payoffs at any two terminal nodes, the SPNE is unique.*

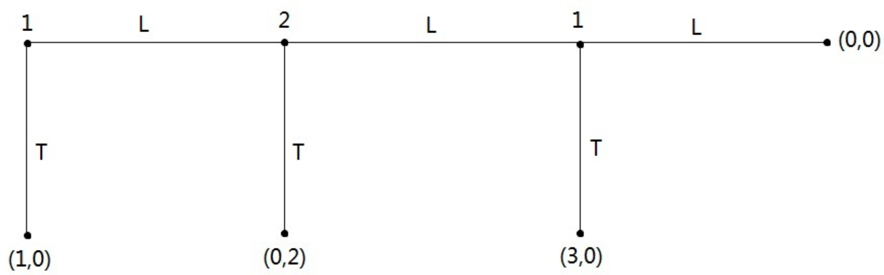
The proof of this proposition simply follows from applying backward induction. One should observe that the notion of SPNE is more restrictive than NE. In fact, for the outcome to be SPNE, we need to assume common knowledge of sequential rationality, rather than only rationality:

[1] For a formal definition of a subgame, please refer to MWG p.274.

Definition 3. *A strategy profile in an extensive form game of perfect information is sequentially rational if each player's payoff is maximized at each decision node, given the other players' strategies.*

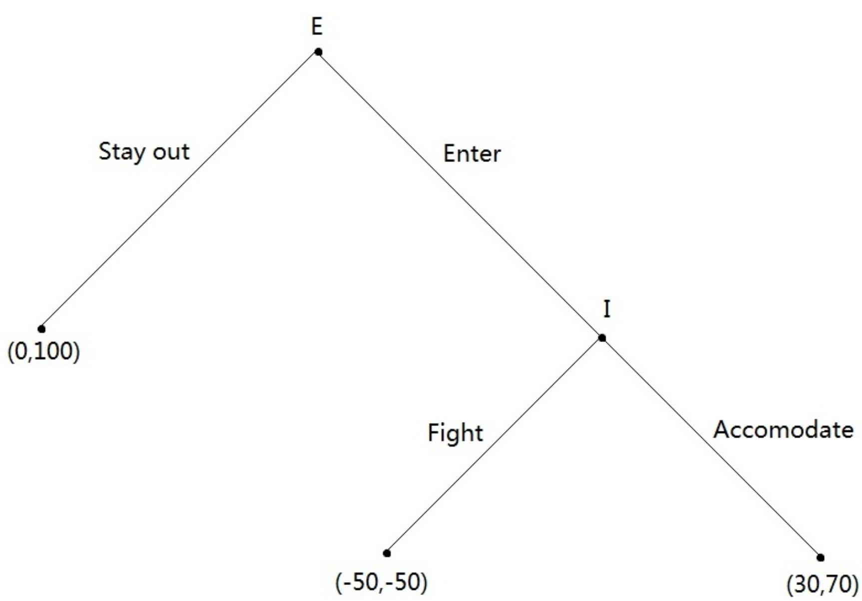
As a result, the set of SPNE can be a proper subset of the set of NE. The following example illustrates one such case.

Example 1. *(Centipede Game) Consider the following game: The unique SPNE is* (TT, T)*.*



Yet there is another NE (TL, T)*.*

Example 2. *(Chain Store Paradox) Consider the following game: The unique SPNE is*



$(Enter, Accomodate)$*. Yet there is another NE* $(Stayout, Fight)$*.*

2 Exercise

2.1 N-firm Cournot Competition

Consider the following Cournot Competition game: the market demand is given by $P = a - bQ$ where Q is the total production level. There are N firms in total with zero marginal cost. Find a symmetric Nash equilibrium.

2.2 Convex Combination of Rationalizable Strategies

In a finite game, assume that A, B are two rationalizable pure strategies for a player. Show that any convex combination of A and B is also rationalizable.

2.3 Location Game

(Micro Comp, Spring 2012) Two restaurants α and β have to simutaneously decide where to locate on a street modeled as the closed interval $[0, 1]$. They can choose any $\frac{k}{n}$ for $k = 0, 1, ..., n$ where $n > 1$ is an even integer. Suppose that customers are uniformly distributed on $[0, 1]$ and each customer will go to the nearest restaurant (ties are broken with equal probability). The objective of each restaurant is to maximize its expected profit, which is proportional to the number of customers it has. Answer the following questions.

(a) Find all rationalizable strategies for each restaurant and all Nash equilibria.

(b) Add one more restaurant γ. Suppose that $n = 8$. Find all rationalizale strategies for each restaurant. Is there a Nash equilibrium?

2.4 Finitely Repeated Prisoners' Dilemma

Suppose that two players play Prisoners' Dilemma for finitely many periods, and the payoff to each agent is the sum of payoffs from every period. Use backward induction to derive a Nash equilibrium. Is there any other NE?